

# From Exact to Partial Dynamical Symmetries: Lessons From the Interacting Boson Model

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We exploit the rich algebraic structure of the interacting boson model to explain the notion of partial dynamical symmetry (PDS), and present a procedure for constructing Hamiltonians with this property. We demonstrate the relevance of PDS to various topics in nuclear spectroscopy, including  $K$ -band splitting, odd-even staggering in the  $\gamma$ -band and anharmonicity of excited vibrational bands. Special emphasis in this construction is paid to the role of higher-order terms.

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## INTRODUCTION

The concept of dynamical symmetry (DS) is now widely accepted to be of central importance in our understanding of many-body systems. In particular, it had a major impact on developments in nuclear [1], molecular [2] and hadronic physics [3], pioneered by F. Iachello and his colleagues. Its basic paradigm is to write the Hamiltonian of the system under consideration in terms of Casimir operators of a set of nested algebras. Its hallmarks are (i) solvability of the complete spectrum, (ii) existence of exact quantum numbers for all eigenstates, and (iii) pre-determined symmetry-based structure of the eigenfunctions, independent of the Hamiltonian's parameters.

The merits of a DS are self-evident. However, in most applications to realistic systems, the predictions of an exact DS are rarely fulfilled and one is compelled to break it. More often one finds that the assumed symmetry is not obeyed uniformly, *i.e.*, is fulfilled by only some states but not by others. The need to address such situations has led to the introduction of partial dynamical symmetries (PDSs) [4]. The essential idea is to relax the stringent conditions of *complete* solvability so that the properties (i)–(iii) are only partially satisfied.

An exact symmetry occurs when the Hamiltonian of the system commutes with all the generators of the symmetry group  $G$ . In this case, all states have good symmetry and are labeled by the irreducible representations (irreps) of  $G$ . The Hamiltonian admits a block structure so that inequivalent irreps do not mix and all eigenstates in the same irrep are degenerate. In making the transition from an exact to a dynamical symmetry, states which are degenerate in the former scheme are split but not mixed in the latter, in accord with the reduction  $G \supset G' \supset G'' \supset \dots$ , and the block structure of the Hamiltonian is retained. Proceeding further to partial dynamical symmetry, some blocks or selected states in a block remain pure, while other states mix and lose the symmetry character. The hierarchy of broken symmetries and the corresponding Hamiltonian matrices are depicted schematically in Fig. 1.

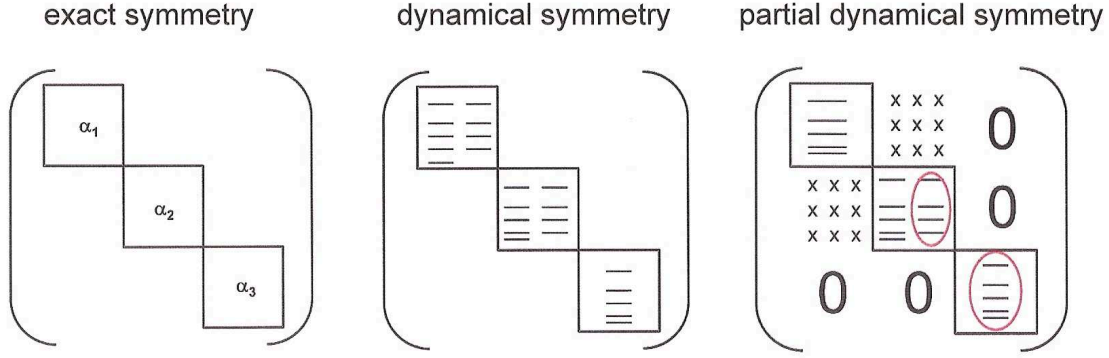


FIG. 1: The Hamiltonian structure in an exact, dynamical (DS) and partial dynamical symmetry (PDS). In an exact symmetry, all states in a given irrep,  $\alpha_i$ , are degenerate. The DS exhibits splitting but no mixing. In a PDS, only selected states (marked by an oval) remain solvable with good symmetry.

### IDENTIFYING HAMILTONIANS WITH PDS

The algorithm for constructing Hamiltonians with PDS [5, 6] starts from the chain of nested algebras

$$\begin{array}{ccccccc} G_{\text{dyn}} & \supset & G & \supset & \cdots & \supset & G_{\text{sym}} \\ \downarrow & & \downarrow & & & & \downarrow \\ [h] & & \langle \Sigma \rangle & & & & \Lambda \end{array}, \quad (1)$$

where, below each algebra, its associated labels of irreps are given. Eq. (1) implies that  $G_{\text{dyn}}$  is the dynamical (spectrum generating) algebra of the system such that operators of all physical observables can be written in terms of its generators; a single irrep of  $G_{\text{dyn}}$  contains all states of relevance in the problem. In contrast,  $G_{\text{sym}}$  is the symmetry algebra and a single of its irreps contains states that are degenerate in energy. For  $N$  identical particles the representation  $[h]$  of the dynamical algebra  $G_{\text{dyn}}$  is either symmetric  $[N]$  (bosons) or antisymmetric  $[1^N]$  (fermions) and will be denoted, in both cases, as  $[h_N]$ . The occurrence of a DS of the type (1) signifies that the Hamiltonian is written in terms of the Casimir operators of the algebras in the chain,  $\hat{H}_{\text{DS}} = \sum_G a_G \hat{C}(G)$ , and the eigenstates can be labeled as  $|[h_N]\langle \Sigma \rangle \dots \Lambda\rangle$ ; additional labels (indicated by  $\dots$ ) are suppressed in the following. The eigenvalues of the Casimir operators in these basis states determine the eigenenergies  $E_{\text{DS}}([h_N]\langle \Sigma \rangle \Lambda)$  of  $\hat{H}_{\text{DS}}$ . Likewise, operators can be classified according to their tensor character under (1) as  $\hat{T}_{[h_n]\langle \sigma \rangle \Lambda}$ .

Of specific interest in the construction of a PDS associated with the reduction (1), are the  $n$ -particle annihilation operators  $\hat{T}$  which satisfy the property

$$\hat{T}_{[h_n]\langle \sigma \rangle \Lambda} |[h_N]\langle \Sigma_0 \rangle \Lambda\rangle = 0, \quad (2)$$

for all possible values of  $\Lambda$  contained in a given irrep  $\langle \Sigma_0 \rangle$  of  $G$ . Any  $n$ -body, number-conserving normal-ordered interaction written in terms of these annihilation operators and their Hermitian conjugates (which transform as the corresponding conjugate irreps) can be added to the Hamiltonian with a DS (1),  $\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + \sum_{\alpha, \beta} A_{\alpha\beta} \hat{T}_{\alpha}^{\dagger} \hat{T}_{\beta}$ , while still preserving the solvability of states with  $\langle \Sigma \rangle = \langle \Sigma_0 \rangle$ . The annihilation condition (2) is satisfied if none of the  $G$  irreps  $\langle \Sigma \rangle$  contained

in the  $G_{\text{dyn}}$  irrep  $[h_{N-n}]$  belongs to the  $G$  Kronecker product  $\langle \sigma \rangle \times \langle \Sigma_0 \rangle$ . So the problem of finding interactions that preserve solvability for part of the states (1) is reduced to carrying out a Kronecker product.

In what follows we illustrate the above procedure and demonstrate the relevance of the PDS notion to nuclear spectroscopy. For that purpose, we employ the interacting boson model (IBM) [1], widely used in the description of low-lying collective states in nuclei in terms of  $N$  interacting monopole ( $s$ ) and quadrupole ( $d$ ) bosons representing valence nucleon pairs. The dynamical algebra is  $G_{\text{dyn}} = \text{U}(6)$  and the symmetry algebra is  $G_{\text{sym}} = \text{O}(3)$ . The Hamiltonian commutes with the total number operator of  $s$ - and  $d$ - bosons,  $\hat{N} = \hat{n}_s + \hat{n}_d$ , which is the linear Casimir of  $\text{U}(6)$ . The model accommodates three DS limits with leading subalgebras  $\text{U}(5)$ ,  $\text{SU}(3)$ , and  $\text{O}(6)$ , corresponding to typical collective spectra observed in nuclei, vibrational, rotational, and  $\gamma$ -unstable, respectively.

### SU(3)-PDS AND AXIALLY DEFORMED NUCLEI

The  $\text{SU}(3)$  DS chain of the IBM and related quantum numbers are given by [1]

$$\begin{array}{ccccc} \text{U}(6) & \supset & \text{SU}(3) & \supset & \text{O}(3) \\ \downarrow & & \downarrow & & \downarrow \\ [N] & & (\lambda, \mu) & K & L \end{array} . \quad (3)$$

The multiplicity label  $K$  is needed for complete classification and corresponds geometrically to the projection of the angular momentum on the symmetry axis. The eigenstates  $|[N](\lambda, \mu)K, L\rangle$  are obtained with a Hamiltonian with  $\text{SU}(3)$  DS which, for one- and two-body interactions, can be transcribed in the form

$$\hat{H}_{\text{DS}} = h_2 \left[ -\hat{C}_2(\text{SU}(3)) + 2\hat{N}(2\hat{N} + 3) \right] + C \hat{C}_2(\text{O}(3)) . \quad (4)$$

Here  $\hat{C}_2(G)$  denotes the quadratic Casimir operator of  $G$ . The spectrum of  $\hat{H}_{\text{DS}}$  is completely solvable with eigenenergies

$$E_{\text{DS}} = h_2 \left[ -(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) + 2N(2N + 3) \right] + CL(L + 1) . \quad (5)$$

The spectrum resembles that of an axially-deformed rotor and the corresponding eigenstates are arranged in  $\text{SU}(3)$  multiplets. In a given  $\text{SU}(3)$  irrep  $(\lambda, \mu)$ , each  $K$ -value is associated with a rotational band and states with the same  $L$ , in different  $K$ -bands, are degenerate. In particular, the lowest  $\text{SU}(3)$  irrep  $(2N, 0)$ , contains the ground band  $g(K = 0)$  and the irrep  $(2N - 4, 2)$  contains degenerate  $\beta(K = 0)$  and  $\gamma(K = 2)$  bands. This  $K$ -band degeneracy is a characteristic feature of the  $\text{SU}(3)$  limit of the IBM which, however, is not commonly observed in deformed nuclei. In the IBM framework, with at most two-body interactions, one is therefore compelled to break the  $\text{SU}(3)$  DS.

To secure solvability and good  $\text{SU}(3)$  symmetry for the ground band, we follow the PDS algorithm and identify  $n$ -boson operators which annihilate all  $L$ -states in the  $\text{SU}(3)$  irrep  $(2N, 0)$ . For that purpose, we consider the following two-boson  $\text{SU}(3)$  tensors,  $B_{[n](\lambda, \mu)\kappa; \ell m}^\dagger$ , with  $n = 2$ ,  $(\lambda, \mu) = (0, 2)$  and angular momentum  $\ell = 0, 2$

$$B_{[2](0,2)0;00}^\dagger \propto P_0^\dagger = d^\dagger \cdot d^\dagger - 2(s^\dagger)^2 , \quad B_{[2](0,2)0;2m}^\dagger \propto P_{2m}^\dagger = 2d_m^\dagger s^\dagger + \sqrt{7} (d^\dagger d^\dagger)_m^{(2)} . \quad (6)$$

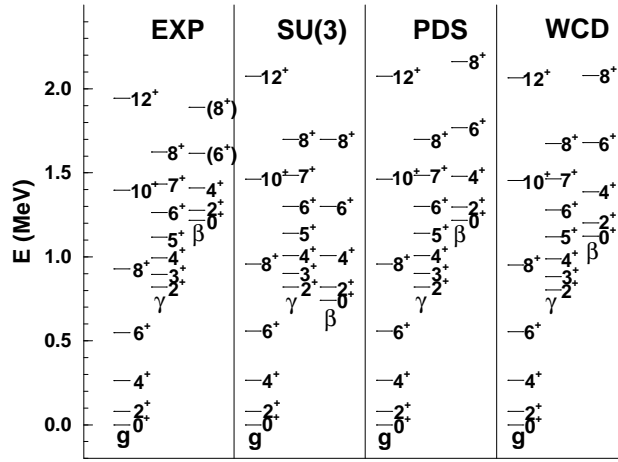


FIG. 2: Spectra of  $^{168}\text{Er}$ . Experimental energies (EXP) are compared with IBM calculations in an exact SU(3) dynamical symmetry [SU(3)], in a broken SU(3) symmetry (WCD) and in a partial dynamical SU(3) symmetry (PDS). The latter employs the Hamiltonian of Eq. (9), with  $N = 16$  and  $h_2 = 4$ ,  $\eta_0 = 4$ ,  $C = 13$  keV [7].

The corresponding annihilation operators,  $P_0$  and  $P_{2m}$ , satisfy

$$\begin{aligned} P_0 |[N](2N, 0)K = 0, L\rangle &= 0, \\ P_{2m} |[N](2N, 0)K = 0, L\rangle &= 0, \quad L = 0, 2, 4, \dots, 2N. \end{aligned} \quad (7)$$

In addition,  $P_0$  satisfies

$$P_0 |[N](2N - 4k, 2k)K = 2k, L\rangle = 0, \quad L = K, K + 1, \dots, (2N - 2k). \quad (8)$$

For  $k > 0$  the indicated  $L$ -states span only part of the SU(3) irreps  $(\lambda, \mu) = (2N - 4k, 2k)$  and form the rotational members of excited  $\gamma^k$  ( $K = 2k$ ) bands. The combination  $P_0^\dagger P_0 + P_2^\dagger \cdot \tilde{P}_2$  is completely solvable in SU(3) and is simply the first ( $h_2$ ) term in  $\hat{H}_{\text{DS}}$ , Eq. (4), related to  $\hat{C}_2(\text{SU}(3))$ . The SU(3)-PDS Hamiltonian is thus given by

$$\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + \eta_0 P_0^\dagger P_0. \quad (9)$$

The relations in Eqs. (7)-(8) ensure that  $\hat{H}_{\text{PDS}}$  retains solvable ground  $g$  ( $K = 0$ ) and  $\gamma^k$  ( $K = 2k$ ) bands with good SU(3) symmetry  $(2N - 4k, 2k)$  and energies as in Eq. (5). The remaining eigenstates of  $\hat{H}_{\text{PDS}}$ , including the  $\beta$  ( $K = 0$ ) band, are mixed.

The SU(3)-PDS spectrum of the Hamiltonian (9) is compared with the empirical spectrum of  $^{168}\text{Er}$  in Fig. 2. As shown, the undesired  $\beta$ - $\gamma$  degeneracy is lifted and the PDS fit is of comparable quality to that of a broken-SU(3) calculation. Since the wave functions of the solvable states (7)-(8) are known, one can obtain *analytic* expressions for matrix elements of observables between them. In particular, the resulting B(E2) values are found to be in excellent agreement with experiment [7], thus confirming the relevance of SU(3)-PDS to the spectroscopy of  $^{168}\text{Er}$ .

The dynamical symmetry expression, Eq. (5), implies a pure rotor spectrum with characteristic  $L(L + 1)$  in-band splitting. Such a pattern is observed in the empirical spectrum of the ground and  $\beta$  bands in  $^{156}\text{Gd}$ , but the  $\gamma$  band exhibits considerable odd-even staggering (OES). As shown in Fig. 3, the empirical staggering index,  $Y(L) = \frac{(2L-1)}{L} \frac{[E(L) - E(L-1)]}{[E(L) - E(L-2)]} - 1$ , displays a pronounced

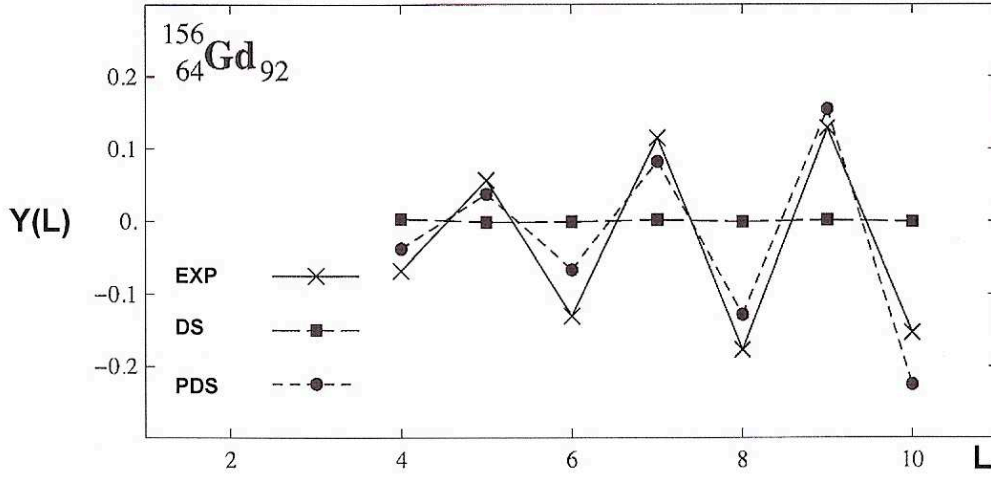


FIG. 3: Experimental (EXP) odd-even staggering in the  $\gamma$ -band of  $^{156}\text{Gd}$ , compared with SU(3)-DS and SU(3)-PDS calculations. The latter employs the Hamiltonian of Eq. (12), with  $N = 12$  and  $h_2 = 7.6$ ,  $C = 10.46$ ,  $\eta_2 = -0.24$ ,  $\eta_3 = 1.68$  keV [8]. The staggering index  $Y(L)$  is defined in the text.

zigzag pattern, in marked deviation from a pure rotor for which  $Y(L) = 0$ . The fact that the SU(3)-DS is obeyed only in selected bands, highlights its partial nature. Following the PDS algorithm, we look for  $n$ -boson operators which annihilate the states in the  $g(K = 0)$  and  $\beta(K = 0)$  bands. For  $n = 3$ , we identify the following SU(3) tensors,  $\hat{B}_{[n](\lambda,\mu)\kappa;\ell m}^\dagger$ , with  $(\lambda, \mu) = (2, 2)$  and  $\ell = 2, 3$

$$\hat{B}_{[3](2,2)2;\ell m}^\dagger \propto W_{\ell m}^\dagger = (P_2^\dagger d^\dagger)_m^{(\ell)} \quad \ell = 2, 3 \quad (10)$$

which satisfy

$$\begin{aligned} W_{\ell m} | [N](2N, 0) K = 0, L \rangle &= 0 \quad L = 0, 2, 4, \dots, 2N \\ W_{\ell m} | [N](2N - 4, 2) K = 0, L \rangle &= 0 \quad L = 0, 2, 4, \dots, (2N - 4) . \end{aligned} \quad (11)$$

The Hamiltonian with SU(3)-PDS, now involving three-body terms, reads [8]

$$\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + \eta_2 W_2^\dagger \cdot \tilde{W}_2 + \eta_3 W_3^\dagger \cdot \tilde{W}_3 . \quad (12)$$

The cubic Casimir operator of SU(3),  $\hat{C}_3(\text{SU}(3))$ , can also be included in  $\hat{H}_{\text{DS}}$ . The relations in Eq. (11) ensure that  $\hat{H}_{\text{PDS}}$  retains solvable ground  $g(K = 0)$  and  $\beta(K = 0)$  bands with good SU(3) symmetry  $(2N, 0)$  and  $(2N - 4, 2)$ , respectively, and energies as in Eq. (5). Other eigenstates, including members of the  $\gamma$  band, are mixed. As seen in Fig. 3, the SU(3)-PDS calculation can adequately reproduce the observed OES in  $^{156}\text{Gd}$  [8]. In this case, the staggering arises from the coupling of the  $\gamma$  band with higher excited bands. Other approaches advocating the coupling of the  $\gamma$  band to the  $\beta$  band [9] or to the ground band [10], cannot describe the OES in this nucleus.

# O(6)-PDS AND $\gamma$ -UNSTABLE NUCLEI

The O(6) DS chain of the IBM and related quantum numbers are given by [1]

$$\begin{array}{ccccccc} \text{U}(6) & \supset & \text{O}(6) & \supset & \text{O}(5) & \supset & \text{O}(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [N] & & \langle \Sigma \rangle & & (\tau) & n_{\Delta} & L \end{array} . \quad (13)$$

The multiplicity label  $n_{\Delta}$  is needed for complete classification. A completely solvable spectrum with eigenstates  $|[N]\langle \Sigma \rangle(\tau)n_{\Delta}L\rangle$  and eigenenergies  $E_{\text{DS}}$ , is obtained with a Hamiltonian with O(6) DS, which has the form

$$\begin{aligned} \hat{H}_{\text{DS}} &= h_0 \left[ -\hat{C}_2(\text{O}(6)) + \hat{N}(\hat{N} + 4) \right] + B \hat{C}_2(\text{O}(5)) + C \hat{C}_2(\text{O}(3)) , \\ E_{\text{DS}} &= h_0 [-\Sigma(\Sigma + 4) + N(N + 4)] + B \tau(\tau + 3) + C L(L + 1) . \end{aligned} \quad (14)$$

The spectrum resembles that of a  $\gamma$ -unstable deformed rotor, where the states are arranged in bands with O(6) quantum number  $\Sigma = N - 2v$ , ( $v = 0, 1, 2, \dots$ ). The ground band ( $v = 0$ ) corresponds to the O(6) irrep with  $\Sigma = N$ . The O(5) and O(3) terms in  $\hat{H}_{\text{DS}}$  (14), govern the in-band rotational splitting.

The O(6)-DS limit provides a good description of the empirical spectrum and E2 rates in  $^{196}\text{Pt}$ , for states in the ground band ( $\Sigma = N$ ). This observation was the basis of the claim [11] that the O(6)-DS is manifested empirically in this nucleus. However, as shown in Fig. 4, the resulting fit to energies of excited bands is quite poor. The  $0_1^+$ ,  $0_3^+$ , and  $0_4^+$  levels of  $^{196}\text{Pt}$  at excitation energies 0, 1403, 1823 keV, respectively, are identified as the bandhead states of the ground ( $v = 0$ ), first- ( $v = 1$ ) and second- ( $v = 2$ ) excited vibrational bands [11]. Their empirical anharmonicity, defined by the ratio  $R = E(v = 2)/E(v = 1) - 2$ , is found to be  $R = -0.70$ . In the O(6)-DS limit these bandhead states have  $\tau = L = 0$  and  $\Sigma = N, N - 2, N - 4$ , respectively. The anharmonicity  $R = -2/(N + 1)$ , as calculated from Eq. (14), is fixed by  $N$ . For  $N = 6$ , which is the appropriate boson number for  $^{196}\text{Pt}$ , the O(6)-DS value is  $R = -0.29$ , which is in marked disagreement with the empirical value. Large anharmonicities can be incorporated in the IBM only by the inclusion of at least cubic terms in the Hamiltonian [12]. One is therefore confronted with the need to select suitable higher-order terms that can break the DS in excited bands but preserve it in the ground band. These are precisely the defining properties of a PDS. Following the general algorithm, we look for  $n$ -boson operators which annihilate all the states in the O(6) ground band. For  $n = 3$ , there are two such O(6) tensors,  $\hat{B}_{[n]\langle \sigma \rangle(\tau)n_{\Delta};\ell m}^{\dagger}$ , with  $\sigma = 1$ ,  $(\tau, \ell) = (0, 0)$  and  $(\tau, \ell) = (1, 2)$ , given by

$$\hat{B}_{[3](1)(0)0;00}^{\dagger} \propto P_0^{\dagger} s^{\dagger} , \quad \hat{B}_{[3](1)(1)0;2m}^{\dagger} \propto P_0^{\dagger} d_m^{\dagger} . \quad (15)$$

Here  $P_0^{\dagger} = d^{\dagger} \cdot d^{\dagger} - (s^{\dagger})^2$ , is an O(6) scalar and  $P_0^{\dagger} P_0$  is simply the first ( $h_0$ ) term in  $\hat{H}_{\text{DS}}$ , Eq. (14), related to  $\hat{C}_2(\text{O}(6))$ . The operators,  $sP_0$  and  $d_m P_0$ , annihilate all  $(\tau, L)$ -states in the O(6) irrep  $\langle \Sigma \rangle = \langle N \rangle$ . This is ensured by the fact that

$$P_0 |[N]\langle N \rangle(\tau)n_{\Delta}L\rangle = 0 , \quad \tau = 0, 1, 2, \dots, N . \quad (16)$$

The only three-body interactions that are partially solvable in O(6) are thus  $P_0^{\dagger} \hat{n}_s P_0$  and  $P_0^{\dagger} \hat{n}_d P_0$ .

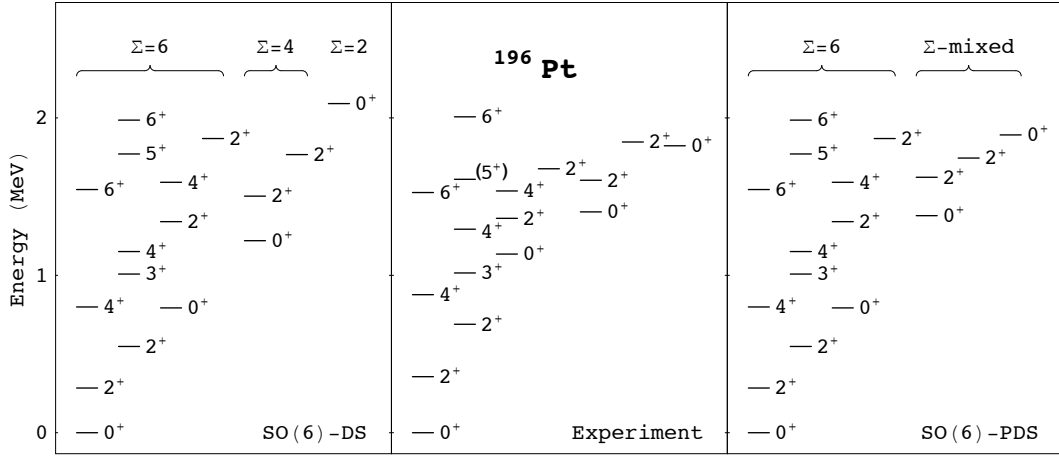


FIG. 4: Observed spectrum of  $^{196}\text{Pt}$  compared with the calculated spectra of  $\hat{H}_{\text{DS}}$  (14), with O(6) dynamical symmetry (DS), and of  $\hat{H}_{\text{PDS}}$  (17) with O(6) partial dynamical symmetry (PDS). The parameters in  $\hat{H}_{\text{DS}}$  ( $\hat{H}_{\text{PDS}}$ ) are  $h_0 = 43.6$  (30.7),  $B = 44.0$  (44.0),  $C = 17.9$  (17.9), and  $\rho_0 = 0$  (8.7) keV. The boson number is  $N = 6$  and  $\Sigma$  is an O(6) label [6].

Since the combination  $P_0^\dagger(\hat{n}_s + \hat{n}_d)P_0 = (\hat{N} - 2)P_0^\dagger P_0$  is completely solvable in O(6), we can transcribe the O(6)-PDS Hamiltonian in the form

$$\hat{H}_{\text{PDS}} = \hat{H}_{\text{DS}} + \rho_0 P_0^\dagger \hat{n}_s P_0. \quad (17)$$

The spectrum of  $\hat{H}_{\text{PDS}}$  (17) is shown in Fig. 4. The states belonging to the  $\Sigma = N = 6$  multiplet remain solvable with energies which obey the same DS expression, Eq. (14). States with  $\Sigma < 6$  are generally admixed but agree better with the data than in the DS calculation. For example, the bandhead states of the first- (second-) excited bands have the O(6) decomposition  $\Sigma = 4$ : 76.5% (19.6%),  $\Sigma = 2$ : 16.1% (18.4%), and  $\Sigma = 0$ : 7.4% (62.0%). Thus, although the ground band is pure, the excited bands exhibit strong O(6) breaking. The calculated O(6)-PDS anharmonicity for these bands is  $R = -0.63$ , much closer to the empirical value,  $R = -0.70$ . It should be emphasized that not only the energies but also the wave functions of the  $\Sigma = N$  states remain unchanged when the Hamiltonian is generalized from DS to PDS. Consequently, the E2 rates for transitions among this class of states are the same in the DS and PDS calculations [6]. Thus, the additional three-body ( $\rho_0$ ) term in the Hamiltonian (17), does not spoil the good O(6)-DS description for this segment of the spectrum.

## SUMMARY AND CONCLUSIONS

The notion of partial dynamical symmetry (PDS) extends and complements the fundamental concepts of exact and dynamical symmetries. It addresses situations in which a prescribed symmetry is neither exact nor completely broken. When found, such intermediate symmetry structure can provide analytic solutions and quantum numbers for a portion of the spectra, thus offering considerable insight into complex dynamics.

On phenomenological grounds, having at hand a concrete algorithm for identifying and constructing Hamiltonians with PDS, is a valuable asset. It provides selection criteria for the a priori

huge number of possible symmetry-breaking terms, accompanied by a rapid proliferation of free-parameters. This is particularly important in complicated environments when many degrees of freedom take part in the dynamics and upon inclusion of higher-order terms in the Hamiltonian. In the IBM examples considered, there are 17 possible three-body interactions, yet only a few terms satisfy the PDS requirements. Furthermore, Hamiltonians with PDS break the dynamical symmetry (DS) but retain selected solvable eigenstates with good symmetry. The advantage of using interactions with a PDS is that they can be introduced, in a controlled manner, without destroying results previously obtained with a DS for a segment of the spectrum. These virtues greatly enhance the scope of applications of algebraic modeling of quantum many-body systems. On a more fundamental level, PDS can offer a possible clue to the deep question of how simple features emerge from complicated dynamics.

PDSs are not restricted to a specific model but can be applied to any quantal systems of interacting particles, bosons, as demonstrated in the present contribution, and fermions [13–15]. They are also relevant to quantum phase transitions [16] and to the study of mixed systems with coexisting regularity and chaos [17, 18].

### ACKNOWLEDGMENTS

It is a pleasure and honor to dedicate this contribution to F. Iachello on the occasion of his 70th birthday. Franco’s innovative approach to physics, emphasizing the deep connection between symmetries and beauty, as manifested in theory and experiment, has influenced and inspired the ideas discussed here. Segments of the reported results were obtained in collaboration with J. E. García-Ramos (Huelva) and P. Van Isacker (GANIL). This work is supported by the Israel Science Foundation.

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